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1996 J. Phys. A: Math. Gen. 29 1929

(http://iopscience.iop.org/0305-4470/29/9/009)

J. Phys. A: Math. Gen. **29** (1996) 1929–1948. Printed in the UK

Phase space geometry and slow dynamics

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Received 18 October 1995

Abstract. We describe a non-Arrhenius mechanism for the slowing down of dynamics that is inherent to the high dimensionality of the phase space. We show that such a mechanism is at work both in a family of mean-field spin-glass models without any domain structure and in the case of ferromagnetic domain growth. The marginality of spin-glass dynamics, as well as the existence of a 'quasi-equilibrium regime' can be understood within this scenario. We discuss the question of ergodicity in an out-of equilibrium situation.

1. Introduction

Many systems of physical interest are out of equilibrium throughout the observation times after preparation. The fact that a system, rather than reaching the Gibbs–Boltzmann equilibrium measure, remains in a regime of slow dynamics can be attributed to various causes. A clear example is the case in which there are domains of different ordered phases growing at the expense of each other, such as when a ferromagnet is quenched to the low-temperature phase. Another rather different scenario is when the phase space has traps of long lifetimes, which the system leaves without visiting again.

Spin-glasses (and also structural glasses) are known to have properties that depend on the 'age' after the quench [1, 2], and hence the possibility that they are in equilibrium is ruled out. Several explanations have been proposed to account for their slow dynamics, based on domain growth ideas [3], on a phase space with traps [4] and on a percolation-like picture in phase space [5, 6]. The latter two scenarios are low dimensional in the sense that they work equally well in a low-dimensional (though infinite) phase space.

The purpose of this paper is to argue, with some examples, that just as equilibrium thermodynamical properties such as the existence of macroscopic non-fluctuating quantities are a direct consequence of the infinite dimensionality of phase space (irrespective of the physical dimensionality D), there are also in the out-of-equilibrium dynamics aspects that are inherent to the geometry of infinite-dimensional (phase) spaces.

We shall first describe these rather generic geometric features, and then show explicitly how they lead to slow dynamics, even in the absence of metastable states. We shall see that they apply to both ferromagnetic domain growth and to a family of mean-field spin-glass models which does not have any domain structure. In both cases we shall concentrate on 'long but finite' times: the limit $N \to \infty$ (or $V \to \infty$) is made before the limit $t \to \infty$.

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0305-4470/96/091929+20$19.50

Here we shall only deal with systems which have an underlying energy landscape, leaving out systems that are defined solely by a dynamical rule.

In order to have a well defined landscape in which a deterministic dynamics takes place, we shall restrict our discussion to zero or near-zero temperatures. We believe that the mechanism we shall describe is also at work in the case of higher temperatures, possibly co-operating with other specifically non-zero temperature mechanisms such as barrier crossing. The three models we shall use as examples have the property that their dynamics at zero and at low temperature are essentially the same.

The problem with extending our geometrical discussion to finite temperatures is that at present we do not know exactly the geometry of *what* we should look into (that is, short of the whole Hilbert space of the Fokker–Planck equation). In this respect, it is normal to have in mind a *free*-energy landscape in terms of variables representing the evolution of a probability packet. Whatever the procedure for the construction of such a landscape, the implicit assumption is that the dynamics is *deterministic* in these variables (otherwise the original energy landscape would be as good).

However, there seems to be discouraging evidence for this approach, at least for glassy systems: it has been shown [7] that a set of trajectories that are forced to coincide up to any given finite time, and are then subjected to different thermal noises, will eventually diverge to distant places of the phase space, while with a deterministic approach one would conclude that they evolve together. In other words, a probability packet that is out of equilibrium is destroyed by the evolution.

We shall concentrate on systems with a smooth energy-density landscape with no relevant infinite energy density configurations. Let us define the normalized square phase-space distance between two configurations s_i^a, s_i^b:

$$B(a,b) = \frac{1}{N}\sum_{i=1}^{N}(s_i^a - s_i^b)^2 \tag{1}$$

or, for two fields $\phi^a(x)$, $\phi^b(x)$:

$$B(a,b) = \frac{1}{V}\int_0^L \mathrm{d}^D x\,(\phi^a(x) - \phi^b(x))^2\,. \tag{2}$$

The correlation function is introduced in the usual way,

$$B(a,b) = C(a,a) + C(b,b) - 2C(a,b)\,. \tag{3}$$

We shall say that a system has well separated energy minima if $B(a,b)$ between any two minima is an O(1) quantity, or

$$\frac{C(a,b)}{[C(a,a)C(b,b)]^{1/2}} < 1\,. \tag{4}$$

In the case of non-zero temperature the corresponding question is whether the correlation between magnetizations in two states is smaller than one:

$$\frac{\sum_i m_i^a m_i^b}{\sqrt{\sum_i (m_i^a)^2}\sqrt{\sum_i (m_i^b)^2}} < 1\,. \tag{5}$$

Some examples of systems with well separated minima are the ferromagnet and ferromagnetic Potts models in any dimension and mean-field spin-glasses with a finite number of breakings. Instead, mean-field spin-glasses with infinitely many levels of replica-symmetry breaking do not satisfy this condition. Our discussion is mainly directed at systems of the first kind.

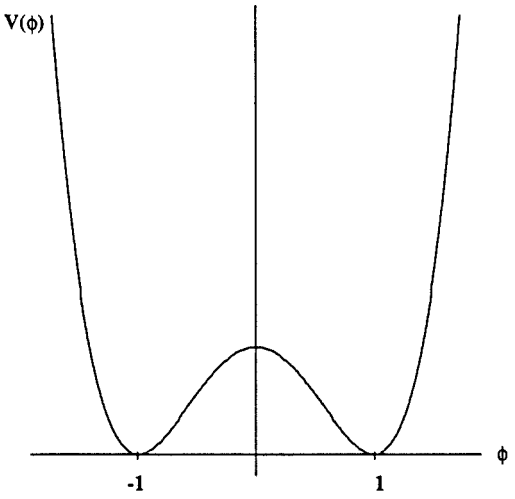

Figure 1. Domain growth potential $V(\phi)$.

We shall exemplify the geometrical properties we discuss here with three models. The first one is a ferromagnetic domain growth problem (see [8] for a review). The energy is of the Landau type

$$E(\phi) = \int \mathrm{d}^D x \left(\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right) \tag{6}$$

where V has a double-well structure with minima at $\phi = \pm 1$ (figure 1). We take $V(\pm 1) = 0$, and consider a system of side L and periodic boundary conditions. In order not to have regions of phase space with infinite energy densities we shall assume that there is an ultraviolet cut-off. The dynamics is gradient descent:

$$\frac{\partial\phi}{\partial t} = -\frac{\delta E}{\delta\phi(x)} = -\nabla^2\phi(x) - V'(\phi(x)) \tag{7}$$

starting from a random configuration.

Secondly, we shall discuss the spherical version of the Sherrington–Kirkpatrick model [9–11]

$$E(s) = -\tfrac{1}{2}\sum_{ij} J_{ij}s_i s_j \qquad \sum_{i=1}^{N} s_i^2 = N \tag{8}$$

where the J_{ij} are quenched random Gaussian variables with zero mean and variance $1/\sqrt{N}$. This model shares some, but not all [11], of the properties of 'true' mean-field spin-glasses, but has the advantage that it allows for a complete analytical description.

The third model we shall consider is a 'true' spin-glass, in that it has slow dynamics and ageing effects, and its Gibbs measure is given by a (one-step) replica-symmetry breaking Parisi solution. It is the p-spin version [12] of the preceding model $p > 2$:

$$E(s) = -\sum_{i_1<i_2<\cdots<i_p} J_{i_1,\ldots i_p}s_{i_1}\ldots s_{i_p} \qquad \sum_{i=1}^{N} s_i^2 = N \tag{9}$$

where the $J_{i_1,\ldots i_p}$ are quenched random Gaussian variables with zero mean and variance $p!/2N^{(p-1)}$. In the large-N limit one can assume that the sum runs over different indices.

In these two last cases we also consider a Langevin dynamics

$$\frac{\partial s_i}{\partial t} = -\frac{\delta E}{\delta s_i} - z(t)s_i + \eta_i \tag{10}$$

where $z(t)$ is a Lagrange multiplier enforcing the spherical constraint, and $\eta_i(t)$ are random uncorrelated white noises with variance $2T$. We shall deal mostly with the zero-temperature case.

Our strategy will be to show that what rather obviously happens in the first two models also happens in a more hidden way in the third, and hence argue that such mechanisms are also at work in glassy dynamics. This will allow us to understand some puzzling aspects of the ageing regime in this kind of system, such as the existence of a 'quasi-equilibrium' (FDT) regime of times even in a well out of equilibrium situation, and the ubiquity of the so-called 'marginality condition' which allows the use of pseudo-static methods to obtain certain dynamical quantities.

The paper is organized as follows. In section 2 we discuss some geometric properties of an infinite-dimensional phase space, and describe how they may lead to a long-time out-of-equilibrium dynamics. In section 3 we show how these considerations apply to the ferromagnetic domain-growth case. The Hessian in this case corresponds to a Schrödinger problem of 'quantum wires'. In section 4 we review some results of [11] for the spherical Sherrington–Kirkpatrick model. This model is also very similar to the domain growth of the O(N) ferromagnet. A complete study of the topology of phase space is extremely simple for it, and in addition we can get a glimpse of the effect of non-zero temperature.

Section 5 contains the main results of this paper. There we study the p-spin spherical model ($p > 2$), which is 'really glassy', in the sense that its dynamics has an ageing regime with long term memory effects [13] qualitatively close to realistic spin-glasses. The equations of motion are in the high temperature phase *exactly* mode-coupling equations. The Parisi ansatz for the replica solution has breaking of the replica symmetry [12] and the phase space has exponentially many valleys [14, 15]. We shall rederive some results of the analytical solution of [13] on the basis of the present geometrical scenario, and compare them with the static approach.

2. Critical points, basins and borders

Borders. Let us start by describing the structure of the phase space of a system with several valleys. First consider the 'critical' or 'stationary' points in which the gradient of the energy vanishes. The nature of a critical point is given by the number of negative eigenvalues of the energy Hessian, which we shall call the 'index' I of the point. The minima (we assume there are at least two) have index zero, the maxima have index N, and the critical points of index one are the saddle points connecting two minima. We shall consider the rather general situation in which there *are* critical points of every index. We shall denote the 'index density' $i \equiv I/N$, $0 \leqslant i \leqslant 1$.

To each minimum is associated a basin of attraction, defined as the set of points that will flow through gradient descent to it. Consider now the $(N-1)$-dimensional border of a basin, which we shall denote by ∂_1. There may be one or more such borders. Now, a point that is strictly on a border will never leave the border (by definition!). Generically, the trajectory will end in a minimum over ∂_1 of the energy. Such minima over ∂_1 are precisely critical points of index one, the saddles separating two true minima.

Hence, we have that ∂_1 is itself divided into basins of attraction, one for each critical point of index one in it. Now consider the $(N-2)$-dimensional border of one such basin.

We shall label it ∂_2. Again, a system starting in ∂_2 (the border of the border) will never leave it. Repeating the argument for ∂_2, we find that it is divided into basins whose minima are the critical points of index two. In this way we can iterate the argument N times, and define ∂_I, the border of the border of ... (I times), on which the trajectory generically flows to a saddle point of index I.

All this description may seem rather baroque, given that most points are not on borders. However, when we consider an infinite-dimensional *phase* space, the structure of borders becomes relevant for the following reasons. A random starting point will be contained within a basin. Now, since such a basin is an N-dimensional object, we know that generically most of its volume is contained within $B \simeq 1/N$ of its border ∂_1 [16]. This in turn means that for $N = \infty$ if the potential is smooth enough the system never leaves the vicinity of ∂_1 in finite times.

The random point being almost on ∂_1, we can repeat the argument to find that it will also be very close to a certain $\partial_2, \ldots$ etc. We can now iterate this argument a *finite* number of times, to find that the system is near a sequence $\partial_1, \ldots, \partial_I$.

We can now understand the origin of the slowing down of the dynamics: a system starting strictly on ∂_I will end up by being stuck in a critical point of index I. A system starting *near* ∂_I will be almost, but not completely stuck, and it slows down. For long times we have that the trajectory manages to distance itself from ∂_I corresponding to degrees of 'bordism' I that are smaller and smaller but still $i = \mathrm{O}(1)$ never distancing itself from ∂_I corresponding to finite I. In other words, the neighbourhoods of the critical points of $i \simeq 0$ are, for long but finite times, efficient in trapping the system. Figure 2 shows how this would come about in a two-dimensional phase space: points starting near the borders have trajectories that take a long time to reach the minimum (of course, the condition of starting near the border is imposed in two dimensions, while it arises naturally in many).

What we have described is a non-Arrhenius mechanism for ageing which works even at zero temperature, and which does not involve any sudden processes of barrier jumping. This mechanism, as we shall see below, is at work in the case of domain growth; the important question here is that it seems rather generic for systems with well separated minima, whether we are able to identify a spatial structure for them or not.

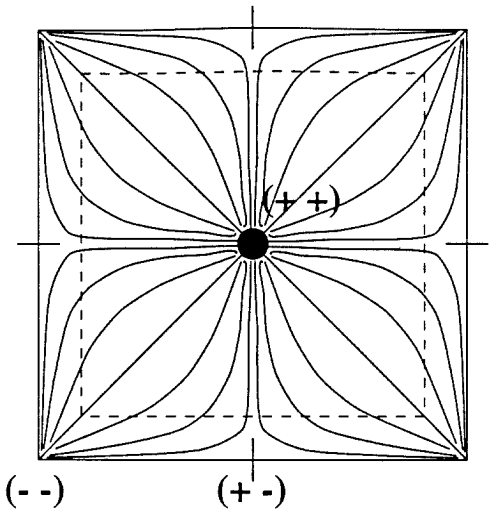

Figure 2. A schematic representation of a basin in 2D phase space. The signs indicate the indices 'I' of the critical points (maxima are vertices and the minimum is at the centre of the square). The trajectories starting near an edge take longer to fall. The broken line represents the 'border' schematically.

Hessian. At long times one knows that the gradient must be small (because the system has slowed down), but is still non-zero so that we are not precisely in a critical point. Indeed, at zero temperature:

$$|\nabla E|^2 = -\frac{\mathrm{d}E(t)}{\mathrm{d}t} \to 0\,. \tag{11}$$

If we now consider the matrix of second derivatives of the energy,

$$\boldsymbol{H}_{ij} = \frac{\partial^2 E}{\partial s_i \partial s_j} \tag{12}$$

we know that in the neighbourhood of a critical point of index I the Hessian has I negative eigenvalues. A natural assumption suggested by the scenario described above is that if we follow a trajectory that starts near ∂_I the spectrum of the Hessian will be similar at every time to that of a nearby critical point, reflecting the degree of 'near bordism' at that time. This is easy to see in figure 2, a trajectory starting near the border will have typically one positive and one negative eigenvalue, until it 'unsticks' from the border and it ends by having two positive eigenvalues.

The dynamics at long times will be such that the H will have a distribution of eigenvalues λ_μ containing $R(t)$ negative eigenvalues, with $R(t)$ decreasing with time, corresponding to a situation in which the landscape at time t is similar to the landscape at a nearby critical point of index $I = R(t)$. The density of the eigenvalues of the H at time t, $\rho_t(\lambda)$ will then contain a bulk of positive eigenvalues, plus a tail extending down to some small negative minimal eigenvalue. The integral over the tail of negative eigenvalues is $R(t)$. The precise manner in which $\rho_t(\lambda)$ tends to its limit $\rho_\infty(\lambda)$ is model-dependent, we shall describe them in detail below for the three models discussed in this paper. The main features are, however, the same: a distribution over positive λ that stabilizes quickly, plus a tail that extends up to negative eigenvalues which tends to disappear slowly with time.

Let us see that the velocity vector points, for long times, in the directions of low (positive and negative) eigenvalues of the Hessian. If there is slower than exponential decay of the energy, we have that

$$\frac{(\nabla E)^+ \boldsymbol{H} (\nabla E)}{|\nabla E|^2} = \frac{\mathrm{d}^2 E}{\mathrm{d}t^2} \Big/ \frac{\mathrm{d}E}{\mathrm{d}t} \to 0\,. \tag{13}$$

Denoting v_μ the component of the velocity in the direction of the eigenvalue λ_μ of $\boldsymbol{H}$, this means that

$$\frac{\sum_\mu v_\mu^2 \lambda_\mu}{\sum_\mu v_\mu^2} \to 0\,. \tag{14}$$

Hence, we have that at long times, the particle moves in a gorge with locally many directions in which it is a minimum, plus a few almost flat directions with positive and negative curvatures: the system is 'critical' or 'marginal' at all finite times. The gradient is small, and is pointing along the almost-flat subspace. We have arrived at this picture by arguing that the dynamics takes place along ridges, and we now find that remarkably, in high dimensions, a ridge can also behave as a channel. It is important to remark that the claim here is not that at long times there should be slow degrees of freedom (this is obvious), but that the existence of such slow directions is a natural consequence of the dominance of borders in high dimensionalities.

Energy differences between critical points, speed of descent. Let us now review a few results that will be useful in the discussion that will follow.

Given two critical points of index I and $I+1$, respectively, which are joined by a gradient line (e.g. a minimum and a saddle), we want to estimate what the energy difference of such a 'step' may be. Consider a 'ladder' of such steps, taking us from a minimum to a saddle, from a saddle to a critical point of index $I=2$, and so on up to a maximum.

If the system does not have infinite energy density configurations, the total energy climbed is $O(N)$, in N steps. Hence, at most a finite number of such steps can be of $O(N)$, and there must be steps of $O(1)$. This means that most of these steps are almost flat.

For D-dimensional systems with short-range interactions we can say more. By considering a domain of a phase A growing against a domain of another phase B, one finds that:

(i) If there is a minimum whose energy is $O(N)$ above another minimum, the energy of the barrier separating them is $O(1)$ [17].

(ii) Between two minima there is at least one saddle (index $I=1$) that is at most $O(N^{1-1/D})$ in energy above them.

Let us now give an upper bound for the time of descent between two points. Consider a point in phase space s^a, and another point s^b which is downhill from s^a along a gradient line. Let their energy difference (energy density difference) be ΔE_{ab} ($\Delta e_{ab} \equiv \Delta E_{ab}/N$), and distance $d_{ab} = |s^a - s^b|$.

We now ask ourselves what is the minimal possible time for descent from a to b. It is easy to show that the time is minimal if the path joining a and b has a constant gradient $= \Delta E_{ab}/d_{ab}$. Hence we have:

$$t_{a\to b} \geqslant \frac{d_{ab}^2}{\Delta E_{ab}} = \frac{B(a,b)}{\Delta e_{ab}}\,. \tag{15}$$

Consider now two minima and their associated saddle point ($I=1$), and let each minimum be well separated from their common saddle $B(\text{saddle, minimum}) > 0$. An immediate consequence of (15) is that if the energy differences between barriers and minima scale with the system size slower than N, the time for descent from the neighbourhood of the saddle to the neighbourhood of either minimum is infinite. This argument certainly holds for finite-dimensional systems with short-range interactions, since for them $\Delta e(\text{barrier, minimum}) \leqslant O(N^{-1/D})$.

Furthermore, since as we have seen before the gradient lines joining most critical points have energy differences of order smaller than N, we find that if two such critical points are well separated, the time of descent from neighbourhoods of each is again infinite.

Ergodicity. If the system has many states, ergodicity is broken in the sense that a single realization of the dynamics will not visit in finite times all configurations of phase space with a probability given by their Gibbs–Boltzmann weights.

However, the fact that in an out-of-equilibrium situation the motion takes place near borders leads to surprises when we attempt to define an 'ergodic component' at a given time. Let us discuss two possible definitions.

Suppose we call the 'ergodic component at time t' the connected set of points that include the configuration $s(t)$ and have an energy lower or equal than $E(t)$ [5], i.e. the set of points to which the system can be driven without work.

Let us now argue that an ergodic component so defined includes at any finite time many, and in systems with finite spatial dimensions, *all* minima. At time t, there are within the ergodic component several points of index $I=1$, having various energies. Each time $E(t)$

reaches the energy of one of these points, the constant-energy surface develops a separatrix and there is a disconnection of a subset of the ergodic component.

It may happen that many (or even all) the critical points of index one are at an energy of order smaller than N above the minima. Indeed, this will always be the case with finite spatial dimensions and short-range interactions. Now, the excess energy $E(t) - E(t = \infty)$ is of $O(N)$ at any finite time. In that case the ergodic component never disconnects and it includes *all* the minima at any finite time t: the dynamics is such that the system refuses to break its ergodicity at finite times.

At non-zero temperature we can discuss ergodicity at a given time from a related but different point of view by asking ourselves whether a configuration at a given time t is doomed to fall in an assigned state, or it may change basins due to thermal fluctuations at times $> t$. That is, we are asking whether the 'target' state is fully determined by the configuration at time t.

We cannot answer this question in general, but in section 4 we will show in a particular model that there is at any given finite time a non-zero probability of changing basin—and this is long before the system has had time to cross barriers between minima. One can suspect that this is quite general, given that at any finite time the system has descended very little from the ridge separating basins (it is close to ∂_1) so the thermal fluctuations may well make it jump across the ridge and head for a different state.

'Quasi-equilibrium' regime and marginality. A rather surprising feature that appears in spin glasses is that if one observes the correlation and response functions at two long but not very separated times, they depend on time differences and obey the fluctuation–dissipation theorem (FDT), just as in a system in equilibrium—even if the system is visiting a region of phase space to which it will never return.

This can be understood within the scenario described above: the fast relaxations are dominated by the local directions with large second derivatives. The form of $\rho_t(\lambda)$ for large times determines the precise time dependence of these relaxations. The slow drift phenomena are related to the motion along the almost flat subspace, i.e. the tail of $\rho(\lambda)$ for λ around zero. The fact that the 'quasi-equilibrium' correlations and response functions depend on time differences reflects the fact that the form of $\rho_t(\lambda)$ for λ well above zero stabilizes quickly, the 'channel walls' in most directions preserve their form.

Another surprising question in mean-field spin-glass dynamics is the so-called 'marginality condition'. In its original form [18], the marginality 'principle' stated that the dynamical values of energy, susceptibility, and the so-called 'anomaly', are determined by the requirement that the fast relaxations (in the FDT regime) be 'critical' or 'marginal', in the sense that they follow power laws instead of exponentials. The dynamics considered there was made manifestly out of equilibrium (though not ageing) by making the Hamiltonian itself (slowly) time-dependent. Because in many models the dynamics in a true equilibrium state is non-critical, the results so obtained differ from those at equilibrium.

It also turned out (although no general proof exists at the moment), that one can obtain the large-time limit of some one-time quantities by solving a static problem and imposing the solution to have marginal stability [18, 13]. The question seems very puzzling: why should the system always choose to fall in a state that is marginal, refusing to see those that are not? Within the present geometrical scenario, the question is quite clear: the dynamics is by construction non-equilibrium, at least at the beginning, even if we always consider a time-independent Hamiltonian. Then we argue that the system never achieves (even local) equilibrium, it does not fall *anywhere*, but is confined near borders of the basins and the

Hessian at long times contains a (decreasing) number of negative eigenvalues. In this sense the dynamics is automatically marginal at all times, whatever the stability of the true minima.

If the minima are $O(N)$ below the borders, we will then observe a finite energy-density difference with respect to them at any finite time. This last thing cannot happen in finite-dimensional systems, but it does happen in the mean-field model we shall discuss in section 5. In that section we shall discuss this question in more detail.

The origin of a 'quasi-equilibrium' regime and the marginality of the long-time dynamics are easy to understand in the case of ordinary domain growth: the response and the correlation function at small time differences are dominated by the bulk of the domains, which are locally (in real space) in equilibrium. The marginality of dynamics is given by the zero modes associated with moving a domain wall. Again, the main point here is that by considering the phase-space geometry we can understand why these things happen in systems which either do not have a real space domain structure, or whose real space structure we do not know.

3. Domain growth

Let us see how the description in the preceding section applies to the case of ferromagnetic domain growth [8], equation (7). For definiteness we restrict ourselves to two dimensions.

We denote the size of the system $V = L^2$ and $L \to \infty$. This case is somewhat complicated by the fact that there is translational invariance, and hence the discussion has to be done modulo translations.

The model has two zero-energy ground states $\phi(x) = \pm 1$, which we depict in figure 3(b) in black and white, respectively. For long times, the system consists of domains of the two types separated by sharp domain walls (figure 3(a)). The energy over the minima is at long times proportional to the total length of all domain walls, which is at finite times $O(L^2)$.

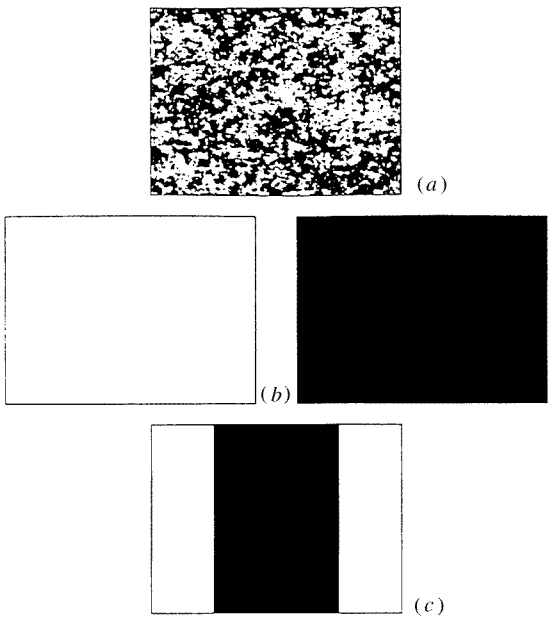

Figure 3. (a) A domain configuration; (b) the two states $\phi = -1$ and $\phi = +1$; (c) a saddle configuration.

The phase-space square distance to the $\pm$ minima is given by

$$B(\phi,\pm)=\frac{1}{V}\left[\int \mathrm{d}^D\phi^2(x)+1\pm 2\int \mathrm{d}^D x\,\phi(x)\right]. \tag{16}$$

At time t the typical domain size is $r(t)$, and by hypothesis we are in the regime $r(t)\ll L$ (figure 3(a)), so that $B(\phi,\pm)$ is $\simeq 2$ at all finite times. The system clearly remains far from either minimum.

The saddles separating minima are easily constructed: they correspond (figure 3(c)) to dividing the volume into two equal pieces of opposite phases, with straight interfaces. There are two continua of such saddles, obtained by translation and 90° rotation. Their energy is $\mathrm{O}(L)$, so that at any finite time the energy of the system is way above the energy required to go from one basin to the other.

The Hessian matrix in the phase-space point $\phi(x)$ is the operator:

$$H=\frac{\delta^2 E}{\delta\phi(x)\delta\phi(y)}=[-\nabla_x^2+V''(\phi(x))]\delta(x-y)\,. \tag{17}$$

The eigenvalues λ_μ and eigenvectors ψ_μ of the Hessian evaluated in ϕ are then obtained from the Schrödinger problem:

$$[-\nabla_x^2+V''(\phi(x))]\psi_\mu(x)=\lambda_\mu\psi_\mu(x) \tag{18}$$

with 'potential energy' given by $V''(\phi(x))$, and periodic boundary conditions for the wavefunctions. The Schrödinger potential is a well that follows the domain walls and rapidly tends to $\sim V''(\pm 1)$ away from them. Figure 4(b) shows the Schrödinger potential across a domain wall.

In order to obtain the complete spectrum, let us first consider a single, straight wall. Translational invariance tells us that

$$\psi_0(x)=\phi'(x) \tag{19}$$

is a bound eigenvector of the Schrödinger potential with $\lambda\sim 0$, which corresponds to shifting the wall. In the saddle-point configuration of figure 3(c) (where two domain walls are present), we have that the $\lambda=0$ eigenvalue is precisely $\phi'(x)$, which is an *odd* function localized near the two domain walls. Since this function has a node, there must be a lower eigenvector which is similarly localized but *even*: its eigenvalue is then negative, and it corresponds to moving the two domain walls in opposite directions.

We can now discuss the structure of the Hessian at large times. The Schrödinger potential consists then of thin wells that follow the domain walls. The structure of bound eigenvalues of such a problem can be appreciated easily by noting that it corresponds to a problem of 'quantum wires' (a 'wire' being the region of each domain wall), a problem of localization that has been extensively studied in the literature [19].

The eigenvectors of H fall into three classes:

(i) All eigenvectors with $\lambda_\mu > V''(\pm 1)$ are unbound. They are simply the bulk oscillations of the magnetizations, and are little affected by the domain structure.

(ii) There are the bound eigenvectors which in the direction perpendicular to the walls of the domains are essentially like (19) (figure 4(c)), and oscillate like $\mathrm{e}^{\mathrm{i}kw}$ in the direction w along the walls. Their eigenvalues are proportional to k^2, and they correspond to the massless spectrum of fluctuations (of length $1/k$) of the domain walls.

(iii) Finally there are negative eigenvalues localized [19] in the more curved regions of the domain wall, with eigenvalues $\lambda\sim -1/r^2$, where r is the local curvature of the domain wall in the region of localization (these are the localized states of the quantum wire problem).

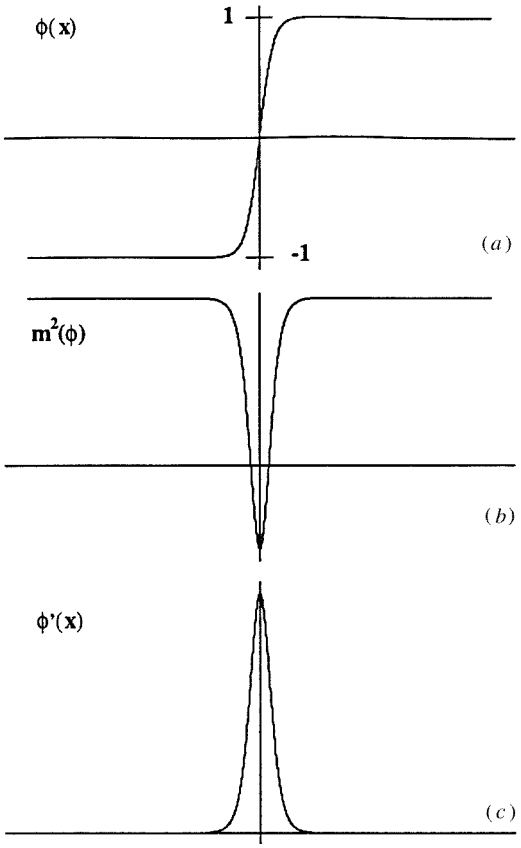

Figure 4. (*a*) $\phi(x)$ across a domain wall; (*b*) The Schrödinger potential $V''(\phi) = m^2(\phi)$ of the corresponding domain wall; (*c*) Schrödinger wavefunction $\psi_\lambda = \phi'(x)$ with $\lambda \sim 0$ across the domain wall.

At any finite time there is then in addition to the 'bulk' of large eigenvalues (i), a tail of small, positive (ii) and negative (iii) eigenvalues. Clearly, as time passes local curvatures become smaller and the domain walls become more and more sparse, so most negative eigenvalues tend to approach zero and simultaneously the distribution of eigenvalues contains fewer and fewer eigenvalues smaller than $\lambda_\mu < V''(\pm 1)$.

The 'velocity vector' $\partial\phi/\partial t$ is in real space concentrated along the regions of fastest variations of ϕ, i.e. in functional space it mainly points in the direction of (ii) and especially (iii).

The fast response is dominated, at large times, by the eigenvalues $\lambda_\mu > V''(\pm 1)$, and coincides with the one obtained at equilibrium.

4. Spherical SK model

The spherical $p = 2$ model has been introduced in [9], where it was shown that its statics has a low-temperature phase with two states. The replica solution does not have replica-symmetry breaking. The long-time dynamics [10, 11] has two regimes of times:

(i) for $t, t' \to \infty$ and $t - t'$ finite the correlation and response functions are time-translational invariant, and satisfy FDT:

$$C(t,t') = C_{\mathrm{FDT}}(t-t')$$
$$R(t,t') = R_{\mathrm{FDT}}(t-t') = \frac{1}{T}\frac{\partial C_{\mathrm{FDT}}(t-t')}{\partial t'} \tag{20}$$

with $C(t,t)=1$. For large $t-t'$, but still $\ll t$,

$$C_{\mathrm{FDT}} \simeq q_{\mathrm{EA}} + A(t-t')^{-1/2} \qquad R_{\mathrm{FDT}} \propto (t-t')^{-3/2} \tag{21}$$

where $q_{\mathrm{EA}} = 1 - T$ is the Edwards–Anderson parameter. At strictly zero temperature $q_{\mathrm{EA}} = 1$, so that in order to see something of the decay of the correlation, and to check FDT, in this regime we will have to go to a low but non-zero temperature.

(ii) The system is not, however in equilibrium at any finite time: In the regime of large, comparable, times in which $t, t' \to \infty$ and $0 < t'/t < 1$ the correlation function is a non-homogeneous function of t'/t, while FDT is violated (see [11]).

Let us see how the considerations of section 3 apply to this case. The energy of the model, defined by the Hamiltonian (8), reads in the basis s_μ in which the matrix J_{ij} is diagonal:

$$E = -\tfrac{1}{2}\sum_\mu J_\mu s_\mu^2 \tag{22}$$

with the spherical constraint

$$\sum_\mu s_\mu^2 = N \tag{23}$$

where J_μ are the eigenvalues of J_{ij}, which for large N are distributed with a semicircle law with support $(-2, 2)$ [20]. Let us denote by $s_1, s_2, \ldots, s_N$ the directions associated with the eigenvalues in decreasing order ($J_1 \simeq 2, \ldots, J_N \simeq -2$).

The stationary points of E, when restricted to the sphere, are the directions of the eigenvectors. There are two minima $s_\mu = \pm\sqrt{N}\delta_{\mu,1}$, two saddle points separating them $s_\mu = \pm\sqrt{N}\delta_{\mu,2}$, and in general two critical points of index I: $s_\mu = \pm\sqrt{N}\delta_{\mu,I}$. The energy difference between the minima and the saddles is easily shown using the semicircle law to be of $\mathrm{O}(N^{1/3})$.

The equation of motion in terms of these variables is, at zero temperature:

$$\frac{\partial s_\mu}{\partial t} = (J_\mu - z(t))s_\mu(t)\,. \tag{24}$$

We assume that the initial configuration $\boldsymbol{s}^0$ is uncorrelated with the potential, and hence in the eigenbasis of J_{ij} each s_μ^0 is a random number of O(1). The solution to the equations of motion then is

$$s_\mu(t) = s_\mu^0 \exp\left[\int_0^t (J_\mu - z(\tau))\,\mathrm{d}\tau\right] \tag{25}$$

where

$$\begin{aligned} z(t) &= \sum_\mu J_\mu s_\mu^2 = -2e(t) < 2 \qquad \forall t \\ z(t) &\simeq 2 - \frac{3}{4t} \qquad \text{as} \quad t \to \infty\,. \end{aligned} \tag{26}$$

From equations (25) and (26) one sees that $s_1(t)$ does not change sign and its absolute value grows steadily. Hence, the two basins of attraction are the set of points:

$$\boldsymbol{s}/s_1 > 0 \qquad \boldsymbol{s}/s_1 < 0\,. \tag{27}$$

The border ∂_1 is then the set $\partial_1 = \{\boldsymbol{s}/s_1 = 0\}$. Repeating again the argument, we conclude that ∂_1 is itself divided into two basins leading to the two saddles. The border between these is $\partial_2 = \{\boldsymbol{s}/s_1 = 0, s_2 = 0\}$. In general $\partial_k = \{\boldsymbol{s}/s_1 = 0, \ldots, s_k = 0\}$.

The normalized squared distance to ∂_1 is $B(\partial_1, \boldsymbol{s}(t)) = s_1(t)^2/N$, and remains of $O(1/N)$ at all finite times. In general, this is true for the distance to ∂_I (I finite), given by

$$B(\partial_I, \boldsymbol{s}(t)) = \frac{1}{N}\sum_{k=1}^{I} s_k(t)^2 \,. \tag{28}$$

We now turn to the study of the Hessian. Because the system is spherically constrained, we have to restrict it to the directions along the sphere. A direct way to see which is the relevant operator is to consider the dynamics of two trajectories that are close to one another: $\boldsymbol{s}(t)$ and $\boldsymbol{s}(t)+\boldsymbol{\sigma}(t)$, with $|\boldsymbol{\sigma}|$ small. Using the equations of motion and the constraint, plus the fact that

$$\boldsymbol{s}(t)\cdot\boldsymbol{\sigma}(t) = 0 \tag{29}$$

we have, in the original basis

$$\frac{\partial\sigma_i}{\partial t} = -H_{ij}\sigma_j - \frac{1}{N}\sum_j \frac{\mathrm{d}s_j}{\mathrm{d}t} s_i\sigma_j \tag{30}$$

where

$$H_{ij} = \frac{\partial^2 E}{\partial s_i \partial s_j} + z(t)\delta_{ij} \tag{31}$$

is the effective Hessian for a spherically constrained system. The last term in (30) serves to impose the preservation of the constraint, and tends to zero with time.

For this model the Hessian reads

$$\boldsymbol{H}_{ij} = -J_{ij} + z(t)\delta_{ij} \,. \tag{32}$$

We note that H is time-dependent (through z), but its eigenbasis does not depend on time.

We now know the structure of eigenvalues of the Hessian for all times: $\rho_t(\lambda)$ is a shifted semicircle law with support in the interval $[-2+z(t), 2+z(t)]$. The distribution is, for large times, a semicircle starting in

$$\lambda_{\min}(t) = -\frac{3}{4t} \tag{33}$$

and extending up to $\lambda \simeq 4$. The number of negative eigenvalues goes as $\sim N t^{-3/2}$. The limiting form of the distribution, $\rho_\infty(\lambda)$, is a semicircle with support in $[0, 4]$.

Let us now discuss how the long-time structure of the Hessian is reflected in the response and correlation functions. Consider the effect of a small kick on the system at time t_w, in the direction of the magnetic field of intensity Δh and duration Δt. It will shift the configuration $s_i(t_w)$ to $s_i(t_w)+\sigma_i(t_w)$, where $\sigma_i(t_w) = \Delta h \Delta t$. The response function at subsequent times is the increase of magnetization due to the field, per unit of $\Delta h \Delta t$, i.e.

$$R(t, t_w) = \frac{1}{\Delta h \Delta t N}\sum_i \sigma_i(t) = \frac{\sum_i \sigma_i(t)\sigma_i(t_w)}{\sum_j \sigma_j(t_w)^2} \,. \tag{34}$$

We can now solve equation (30), neglecting its last term, to get

$$\sigma_i(t) = \sum_j \mathcal{T}\left[\mathrm{e}^{-\int_{t_w}^{t} \boldsymbol{H}(\tau)\,\mathrm{d}\tau}\right]_{ij} \sigma_j(t_w) \tag{35}$$

where $\mathcal{T}$ denotes time order (irrelevant in this case, since the eigenbasis does not evolve). Leading to

$$R(t, t_w) = \frac{1}{\sum_j \sigma_j(t_w)^2}\sum_{ij} \sigma_i(t_w)\mathcal{T}\left[\mathrm{e}^{-\int_{t_w}^{t} \boldsymbol{H}(\tau)\,\mathrm{d}\tau}\right]_{ij}\sigma_j(t_w) \,. \tag{36}$$

Because the eigenbasis of the Hessian is uncorrelated with the direction of the magnetic field, this becomes, in the large-N limit:

$$R(t,t_w) = \frac{1}{N}\,\mathrm{tr}_\perp\,\mathcal{T}\left[\mathrm{e}^{-\int_{t_w}^{t}\boldsymbol{H}(\tau)\,\mathrm{d}\tau}\right] = \mathrm{tr}_\perp\left[\mathrm{e}^{-\langle\boldsymbol{H}\rangle_{t,t_w}(t-t_w)}\right] \tag{37}$$

where the $\mathrm{tr}_\perp$ denotes the trace restricted to the directions tangential to the sphere, and we have defined the time-averaged Hessian as

$$\langle H_{i,j}\rangle_{t,t_w} \equiv \frac{1}{t-t_w}\int_{t_w}^{t} H_{ij}(\tau)\,\mathrm{d}\tau\ . \tag{38}$$

It is clear that in (37) the contribution of the tail of low eigenvalues of the averaged Hessian is negligible for finite time separations $t-t_w \ll t$, and that for these time separations we can substitute the averaged Hessian by the asymptotic (semicircle) distribution $\rho_\infty(\lambda)$. Hence, we find that the fact that $\rho_t(\lambda)$ has a limit implies time homogeneity in this regime of times. Furthermore, the fact that $\rho_\infty(\lambda) \propto \lambda^{1/2}$ for small λ (but large compared with $1/t$) implies that $R(t-t_w) \propto (t-t_w)^{-3/2}$.

For time separations of $t-t_w$ of the order of t, the exponential in (37) selects the tail of lowest eigenvalues, and the FDT regime breaks down: the tail of almost flat directions of $\boldsymbol{H}$ is responsible for the 'ageing regime' $t-t_w = \mathrm{O}(t)$.

For this simple model, the calculation can be carried out explicitly, using the asymptotic form for H at long times:

$$\langle H_{ij}\rangle_{t,t_w}(t-t_w) = -J_{ij}(t-t_w) + \left[\int_{t_w}^{t} z(\tau)\,\mathrm{d}\tau\right]\delta_{ij} = [-J_{ij}+2\delta_{ij}](t-t_w) - \frac{3}{4}\,\delta_{ij}\ln\frac{t}{t_w} \tag{39}$$

from which an expression for the ageing regime can be readily found.

Let us now turn to the 'fast' correlation function at small but non-zero temperature and at two large but not very separated times. Because the motion along the flat directions is slow, it can be neglected for short time differences. At the other extreme, one can assume that the system is equilibrated in the 'fast' degrees of freedom corresponding to large eigenvalues of $\boldsymbol{H}$. Since, as we have seen, in this regime of times it is only such degrees of freedom that also contribute to the response function, we conclude that FDT must hold. The correlation function then reads

$$C(t,t') = 1 - T\int_{t'}^{t} R(t,t'')\,\mathrm{d}\tau = 1 - T\ \mathrm{tr}_\perp\int_{t'}^{t}\mathrm{d}t''\left[\mathrm{e}^{-\int_{t''}^{t}\boldsymbol{H}(\tau)\,\mathrm{d}\tau}\right]. \tag{40}$$

Since $t-t'$ is by assumption $\ll t$, we can neglect the variation of $\boldsymbol{H}$ to get

$$C(t,t') = 1 - T\,\mathrm{tr}_\perp\left\{\boldsymbol{H}^{-1}(t)\left[1-\mathrm{e}^{-\boldsymbol{H}(t)(t-t')}\right]\right\} \simeq 1 - T\sum_\mu \frac{1-\mathrm{e}^{-\lambda_\mu(t-t')}}{\lambda_\mu}\ . \tag{41}$$

We can now check that the assumptions we made above are consistent: for $t-t'$ finite and large t, the numerator in (40) acts as a low λ_μ cut-off: the (few) positive and negative λ_μ that are close to zero $\mathrm{O}(1/t)$ do not contribute. As $t-t'$ becomes comparable with t the approximation (and hence the validity of FDT) breaks down because on the one hand

we can no longer assume the constancy of $\boldsymbol{H}$, and on the other hand the slow degrees of freedom ($\lambda_\mu \sim 0$) start contributing and we cannot assume that they are equilibrated.

The inverse eigenvalue $1/\lambda_\mu$ is the typical length of the fluctuations in the direction μ, it is infinite if $\lambda \leqslant 0$. The Edwards–Anderson parameter is

$$q_{\mathrm{EA}} = \lim_{t-t'\to\infty} \lim_{t\to\infty} C(t,t') \,. \tag{42}$$

The quantity

$$a_q \equiv \frac{1-q_{\mathrm{EA}}}{T} \tag{43}$$

precisely measures the average width of the channel, the order of the limit ensures (via the numerator in (41)) a cut-off in the directions in which the system is unbound, i.e. it selects the 'walls' of the channel against the longitudinal direction. Note that in this model there is *no* discontinuous process of escape from a trap.

Finally, let us discuss the question of ergodicity breaking. As we have noted already, the separation between minima is of the order of $\mathrm{O}(N^{1/3})$, while the energy above the minima is, for large times, $\Delta E(t) \simeq \frac{3}{8t}N$, well above the barrier. In this model we can also ask ourselves about ergodicity in the other sense of section 3: whether there is the possibility, at non-zero temperature, of changing basin spontaneously.

We have seen that it is the sign of s_1 (in the eigenbasis of J_{ij}) which defines the basin. The evolution of s_1 is, for finite temperature T given by (see II.4 of [11])

$$s_1(t) = s_1(t=0)\mathrm{e}^{-\int_0^t \mathrm{d}\tau\,(z(\tau)-2)\,\mathrm{d}\tau} + \int_0^t \mathrm{d}t''\,\mathrm{e}^{-\int_{t''}^t \mathrm{d}\tau\,(z(\tau)-2)}\eta(t'') \,. \tag{44}$$

The first term is deterministic, while the second is a Gaussian random variable with variance:

$$2T\int_0^t \mathrm{d}t''\,\mathrm{e}^{-\int_{t''}^t 2(z(\tau)-2)\,\mathrm{d}\tau} \tag{45}$$

a quantity of order one. Hence, there is at any finite time the probability that $s_1(t)$ will change sign (unless, of course, the system started well within a basin: $s_1(t=0) = \mathrm{O}(\sqrt{N})$

5. *p*-spin model

The spherical p-spin model [12], unlike the previous two, has many (exponentially with N) [15] minima. The Parisi ansatz for the replica solution has a one-step replica-symmetry breaking.

The long-time out-of-equilibrium dynamics [13] has, again, in the low-temperature phase, two regimes of times:

(i) For $t, t' \to \infty$ and $t - t'$ finite the correlation and response functions are time-translational invariant, and satisfy FDT. At *zero temperature* and large $t - t'$, but still $\ll t$, also for this model

$$C_{\mathrm{FDT}} \simeq q_{\mathrm{EA}} + A(t-t')^{-1/2} \qquad R_{\mathrm{FDT}} \propto (t-t')^{-3/2} \,. \tag{46}$$

The relaxation exponents change with temperature and are given in [21].

The dynamical Edwards–Anderson parameter ($T \sim 0$)

$$\frac{1-q_{\mathrm{EA}}}{T} = \sqrt{\frac{2}{p(p-1)}} \tag{47}$$

as well as the asymptotic 'threshold' energy density

$$e_{\text{thres}} = -\sqrt{\frac{2(p-1)}{p}} \tag{48}$$

are *different* from their corresponding Gibbs-measure counterparts. We shall re-derive them from a geometrical point of view below.

(ii) For widely separated times, the correlation becomes smaller than q_{EA}, and is not time-translational invariant. The response function in this regime also violates FDT, and, unlike the previous case, yields a long-time memory with ageing effects that are similar to those experimentally observed in real spin-glasses.

Structure of minima. The system has many minima, ranging in energy density from the ground state e_{Gibbs} (as calculated in the replica calculation) up to a threshold energy e_{thres} [13–15] (48). The corresponding Edwards–Anderson parameters of these states are given, in terms of the energy density of each minimum e, by [14]

$$\frac{1-q_{\text{EA}}}{T} = \frac{1}{p-1}\left\{-e - (e^2 - e_{\text{thres}}^2)^{1/2}\right\}. \tag{49}$$

The asymptotic energy, as well as the dynamical Edwards–Anderson parameter, tend to the values (48), (47) corresponding to the threshold states, though *the system never relaxes into any of these* [13].

Using the equation of motion and the spherical constraint, one has that the Lagrange multiplier $z(t)$ is related to the energy density by

$$z(t) = -pe(t). \tag{50}$$

The energy Hessian can be calculated by expanding up to second order around a stationary point and using the spherical constraint, to get (cf equation (31))

$$H_{ij} = \frac{\partial^2 E}{\partial s_i \partial s_j} + z(t)\delta_{ij}. \tag{51}$$

Using the homogeneity in the expression for the energy, we first note that the equation of stationarity implies that $\boldsymbol{H}$ has an eigenvector in the 'radial' direction s_i of eigenvalue $-p(p-1)z(t)$.

We can find the spectrum in the directions orthogonal to this one by using the 'locator expansion' [22], or directly by noting that $\partial^2 E/\partial s_i \partial s_j$ is a sum of many ($\sim N^{p-2}$) terms, and hence assuming that in the directions orthogonal to $\boldsymbol{s}$ the couplings can be taken as uncorrelated from the configurations in the large-N limit (this is indeed the assumption in [22]).

The mean-squared element can be obtained from:

$$\begin{aligned}\overline{\left[\frac{\partial^2 E}{\partial s_i \partial s_j}\right]^2} &= \left[\frac{p(p-1)}{p!}\right]^2 \sum_{i_1,\ldots,i_{p-2}\neq i,j} \sum_{j_1,\ldots,j_{p-2}\neq i,j} \overline{J_{i_1,\ldots,i_{p-2}} J_{j_1,\ldots,j_{p-2}}} s_{i_1}\cdots s_{i_{p-2}} s_{j_1}\cdots s_{j_{p-2}} \\ &= \left[\frac{p(p-1)}{p!}\right]^2 N^{p-2}\left(\frac{p!}{2N^{p-1}}\right)(p-2)!\frac{p(p-1)}{2N}\end{aligned} \tag{52}$$

where we have used the variance of the couplings, and the factor $(p-2)!$ counts the number of ways of matching the $\{i_k\}$ with the $\{j_k\}$. The matrix of second derivatives is then a random matrix whose distribution of eigenvalues is a semicircle law with support in $[-\sqrt{2p(p-1)}, +\sqrt{2p(p-1)}]$, plus a projector in the direction of $\boldsymbol{s}$.

Using equations (50)–(52) and (48) we find that the spectrum of the Hessian $\rho_e(\lambda)$ in the direction tangential to the constraint is a shifted semicircle law with support in

$$p(e_{\rm thres} - e) < \lambda < -p(e_{\rm thres} + e) \qquad \lambda_{\rm min} = p(e_{\rm thres} - e)\,. \tag{53}$$

A direct calculation using

$$\frac{1 - q_{\rm EA}}{T} = \int {\rm d}\lambda \frac{\rho_e(\lambda)}{\lambda} \tag{54}$$

yields back (49). A more complete confirmation of (53) can be obtained by making a low-temperature expansion around a minimum and checking that using the form of $\rho_e(\lambda)$ one gets the same results as in the TAP approach of [14].

We now understand the origin of the threshold level: the 'gap' in the spectrum of the Hessian becomes smaller as one considers states that are higher, until it disappears at $e = e_{\rm thres}$. Above the threshold the gap is negative: the critical points are unstable and their index increases with increasing energy.

Finally, we can estimate the energy difference between the critical points of index I and the highest threshold minima. Using $\lambda_{\rm min}$ from (53) and the semicircle law one easily gets

$$\begin{aligned} &I = N \int_{\lambda_{\rm min}}^{0} {\rm d}\lambda\, \rho_I(\lambda) \simeq N \int_0^{p(e_I - e_{\rm thres})} {\rm d}\lambda\, \lambda^{1/2} \\ &I \propto N(e_I - e_{\rm thres})^{3/2} \Rightarrow E_I - E_{\rm thres} \propto I^{2/3} N^{1/3}\,. \end{aligned} \tag{55}$$

In particular, we have for the barriers separating threshold minima ($I = 1$):

$$E_{I=1} - E_{\rm thres} \sim N^{1/3}\,. \tag{56}$$

Out-of-equilibrium dynamics. In order to study the long-time dynamics starting from a random configuration, we start by considering the Hessian given by (30) and (31). The calculation of the *time-dependent* spectrum of the second derivatives of the energy is obtained by making, as in the preceding section, the assumption of independence of the configurations and couplings in the large-N limit. Repeating the calculation (51) for this case, we find that the Hessian consists of a random matrix of elements with variance as in (52) plus a shift term. The eigenvalue density $\rho_t(\lambda)$ is then a semicircle law with support in

$$p(e_{\rm thres} - e(t)) < \lambda < -p(e_{\rm thres} + e(t)) \qquad \lambda_{\rm min}(t) = p(e_{\rm thres} - e(t))\,. \tag{57}$$

This assumption is confirmed numerically in figure 5, where we plot the integrated spectrum of the Hessian for different times. In the inset we show the integrated spectrum of the matrix of second derivatives of the energy (i.e. the spectrum of $\boldsymbol{H}$ minus the shift) and compare it with an integrated semicircle law.

We are now in a position to rederive some results for the out-of-equilibrium dynamics of [13]: if we now claim, as in the preceding sections, that because of the dominance of borders the dynamics is such that the Hessian has a (decreasing) number of negative eigenvalues at all finite times, we reobtain the 'marginality condition':

$$\lim_{t \to \infty} e(t) = e_{\rm thres}\,. \tag{58}$$

At this point it is important to remark that this last equation does *not* mean that the system relaxes into a near-threshold state: at all finite times an infinite system has a Hessian with an *infinite* number of directions in which the energy is a maximum. If at a given finite time the system is close to a border ∂_I, we may ask how many different basins meet there,

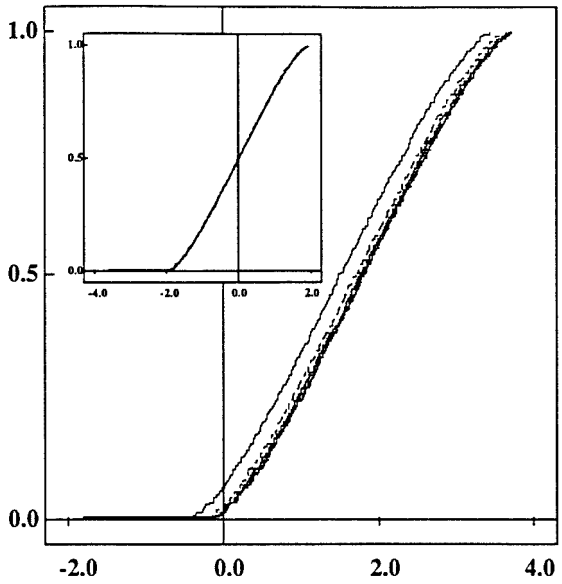

Figure 5. Integrated eigenvalue ($\lambda/\sqrt{3}$) distribution of the Hessian at 10 different times for $p = 3$ and $N = 200$. Inset: the shifted distribution for all times and the analytical integrated semicircle law.

i.e. to what extent the system is 'almost undecided'. A study [23] of the random partitioning of a high-dimensional space suggests that generically $I + 1$ basins meet at ∂_I—an infinite quantity at finite times.

We have seen that the saddles separating threshold minima are typically $O(N^{1/3})$ above the threshold level, while the energy is at all finite times $O(N)$ above this level. Again, we confirm that at all finite times the constant-energy surface is *not* disconnected into components.

Up to now the system seems to behave dynamically in a very similar fashion to the $p = 2$ model. An important difference appears when we consider the evolution of the eigenbasis of the Hessian. Because the actual elements of the second derivative matrix now depend on the time via the time dependence of the spins, we may expect that unlike the case $p = 2$ the eigenbasis of H also depends on time. Indeed (see figure 6) the overlap of the eigenvectors at two different times $(t, t'$ such that $C(t, t') \sim 0.7)$ is very small. Although the spectrum of the Hessian leads us to an image of a 'channel' whose characteristics change slowly with time (as in the case $p = 2$), we now see that such a channel twists and turns chaotically with time.

Let us now turn to the analysis of the 'fast' relaxations (the quasi-equilibrium regime). We have seen in the preceding section that what is relevant for this regime is the spectrum of the time-averaged Hessian (cf equations (37), (38) and (41)). Since for this problem the eigenbasis of the Hessian turns, in principle we have to take care of the fact that the spectrum of the time-averaged Hessian is not the same as the time average of the spectrum.

The spectrum of the averaged Hessian can still be obtained with the same assumptions, but taking into account in (52) that the spins are evaluated at two different times. This will introduce a factor $\sim C(t, t')^{p-2}$. However, since here we are interested in the regime of times for which $C(t, t') > q_{\mathrm{EA}}$, and we are restricting ourselves to *near-zero temperatures*, the effect of the variations of the configuration (and hence the 'turning' and the time ordering

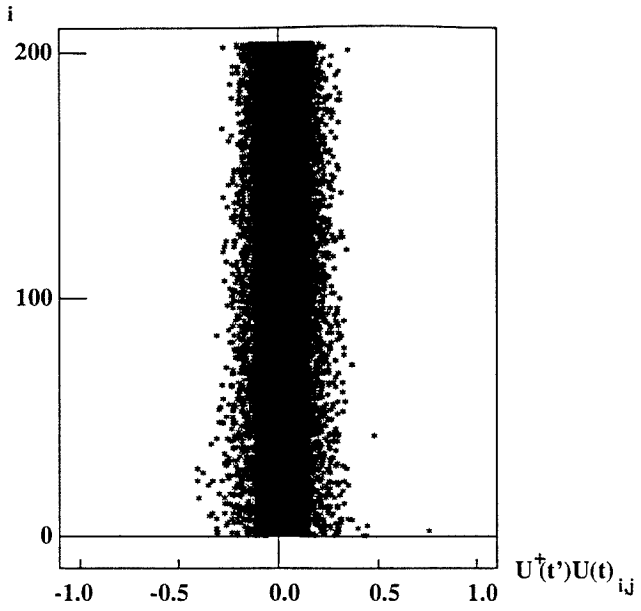

Figure 6. The statistics of overlaps between the eigenvectors of the Hessian, at two times ($[U^{\dagger}(t')U(t)]_{i,j}$ versus i), for $p = 3$ and $N = 200$.

of the Hessian) can be neglected. Following a calculation as in the preceding section, we reobtain (46).

Hence, we see that the fact that the steepest 'walls' of the 'channel' preserve their form (even if the channel twists) explains the validity of the fluctuation–dissipation theorem and time-translational invariance in the quasi-equilibrium regime.

6. Conclusions

In this paper we have argued that the relaxational dynamics of a system in the thermodynamic limit has characteristics that lead naturally to slowing down and are a direct consequence of the infinite-dimensionality of phase space.

The peculiarities of infinite-dimensional geometry may be overlooked if one seeks inspiration from a low-dimensional phase-space sketch: such a sketch for a ferromagnet would consist of a double well and would lead us to conclude that after a rapid quench the ferromagnet 'falls' into one of the states, though we know that this does not happen in finite times.

In the case of a ferromagnet, we are kept from jumping to wrong conclusions by a picture of snapshots of the domain structure at different times—a naturally infinite-dimensional description of a configuration. At present it is not known if every form of slow dynamics present in nature is just some sophisticated version of domain growth (cf the long-lasting controversy with spin-glasses): hence the interest of trying to explore the consequences of infinite-dimensionality of phase space directly, without invoking real-space structures.

In order to isolate these phase-space geometric causes of slowing down (which do not involve rapid jumps) from barrier-crossing mechanisms, we have deliberately concentrated on systems which have a non-Arrhenius behaviour at near-zero temperatures. At the other extreme, there is a picture by Bouchaud [4] of jumps between phase space traps which is at the same time simple and yields excellent results for spin-glasses.

Whichever turns out to be the complete description of the problem of slow dynamics, it

will have to take into account all elements, blending continuous and discontinuous trapping together in a single picture—possibly with each correlation scale dominated by one type of mechanism.

Let us finally remark that though it seems important to have a phase-space intuition of what happens with a system that ages, it is not a substitute for dynamical computation: by the time one introduces all the necessary elements that go into the definition of a 'barrier', one has taken into account all the paths leading to it and their respective probabilities: precisely what one computes in a dynamical calculation. Moreover, if as we have argued here the saddle points are also relevant, in order to know their structure we are faced with a Morse-theory problem that is again best studied [24] using methods that are closely related to Langevin dynamics.

Acknowledgments

We wish to thank J P Bouchaud, L F Cugliandolo, P Le Doussal and M Mezard for discussions and suggestions.

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